

Minimal Lagrangian tori in Kahler-Einstein manifolds

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Abstract

In this paper we use structure preserving torus actions on Kahler-Einstein manifolds to construct minimal Lagrangian submanifolds. Our main result is: Let N^{2n} be a Kahler-Einstein manifold with positive scalar curvature with an effective T^n -action. Then precisely one regular orbit L of the T -action is a minimal Lagrangian submanifold of N . Moreover there is an $(n-1)$ -torus $T^{n-1} \subset T^n$ and a sequence of non-flat immersed minimal Lagrangian tori L_k , invariant under T^{n-1} s.t. L_k locally converge to L (in particular the supremum of the sectional curvatures of L_k and the distance between L and L_k go to 0 as $k \mapsto \infty$).

1 Introduction

In this paper we will use torus actions on Kahler-Einstein manifolds with positive scalar curvature to construct minimal Lagrangian tori.

Let N^{2n} be a Kahler-Einstein (K-E) manifold with positive scalar curvature and suppose we have a structure preserving T^k -action on N . We will look for T -invariant minimal Lagrangian submanifolds of N . If $k = n$ then we have shown in [4] that there is precisely one regular orbit of the T -action, which is a minimal Lagrangian submanifold of N . In this paper we study the case when $k = n - 1$ (a complexity one action). The main tool in our investigation will be a correspondence between minimal Lagrangian submanifolds of N and certain Special Lagrangian submanifolds of $K(N)$ - the total space of the canonical bundle of N .

The manifold $K(N)$ has a natural holomorphic volume form φ . Also since N is K-E with positive scalar curvature, we have a (Calabi) metric ω_u on $K(N)$, which is a Ricci-flat Kahler metric (see Section 2.1). The form φ is covariantly constant with respect to ω_u , and we have Special Lagrangian (SLag) submanifolds $L' \subset K(N)$, defined by the conditions $\omega_u|_{L'} = 0$ and $Im\varphi|_{L'} = 0$ (see [5] and [4]). There is a radial vector field Y on $K(N)$, whose flow is scaling of $K(N)$ by real numbers (see Section 2.1). Our main tool in studying minimal Lagrangian submanifolds on N will be a correspondence between minimal La-

grangian submanifolds on N and SLAG submanifolds on $K(N)$, invariant under the flow of Y (see Lemmas 1 and 2 in Section 2.1).

In Section 2.2 we study SLAG submanifolds on $K(N)$ using a torus action on N . Suppose we have a T^k -action on N . This action of course induces a T^k -action on $K(N)$. We will see that there are canonical moment maps μ on N and μ' on $K(N)$. Let $Z \subset N$ be the zero set of μ and $\pi : K(N) \mapsto N$ be the projection. Then the zero set of μ' is $Z' = \pi^{-1}(Z)$. Suppose that T acts freely on $Z'' = Z' - Z$. Then we have a symplectic reduction $Q = Z''/T$. We will see that Q has a natural holomorphic volume form φ' and a metric ω' and SLAG submanifolds of (Q, φ', ω') lift to T -invariant SLAG submanifolds of $K(N)$. Also the vector field Y is tangent to Z'' and projects to a vector field Y' on Q . Thus we reduced the problem of finding minimal Lagrangian submanifolds of N to a problem of finding SLAG submanifolds of Q , invariant under the flow of Y' .

In Section 2.3 we assume that $k = n - 1$. Let $X \subset Z''$ be the set of elements of Z'' of unit length in $K(N)$ and $S = X/T \subset Q$. We will see that there is a non-vanishing vector field W on S s.t. there is a correspondence between Y' -invariant SLAG submanifolds of Q and trajectories of the W -flow on S .

Next we would like to develop a criterion to see that T^{n-1} acts freely on Z'' . We also would like to understand periodic orbits of the vector field W on S (to construct immersed minimal Lagrangian tori on N). We can do it if we assume that N is a toric K-E manifold (see Section 3.1). In this case we can prove the following Theorem:

Theorem 1 *Let N^{2n} be a K-E manifold with positive scalar curvature with an effective T^n -action. Then precisely one regular orbit L of the T -action is a minimal Lagrangian submanifold of N . Moreover there is an $(n - 1)$ -torus $T^{n-1} \subset T^n$ and a sequence of non-flat immersed T^{n-1} -invariant minimal Lagrangian tori $L_k \subset N$ s.t. L_k locally converge to L (in particular the supremum of sectional curvatures of L_k and the distance between L and L_k go to 0 as $k \mapsto \infty$).*

Here by local convergence we mean the following: The distance between L_k and L goes to 0 as $k \mapsto \infty$. Also for any point $l \in L$ we can choose a neighbourhood U of l in N s.t. $L_k \cap U$ is a finite union L_k^j of submanifolds of the form $L_k^j = \exp(v_k^j)(L \cap U)$, where v_k^j is a normal vector field to L on $L \cap U$. Moreover any subsequence v_k^j converges to 0 in a C^∞ topology as $k \mapsto \infty$.

This result is new even for $N = \mathbb{C}P^n$ for $n \geq 3$. For $n = 2$ examples of non-flat S^1 -invariant immersed minimal Lagrangian tori in $\mathbb{C}P^2$ were constructed in [3] and [6] using harmonic maps.

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In this paper we use a number of results from our previous paper [4], including proofs for the completeness of exposition.

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2 Minimal Lagrangian submanifolds in complex one K-E manifolds

2.1 A correspondence between minimal and Special Lagrangian submanifolds

Let N^{2n} be a K-E manifold with positive scalar curvature. We begin by reviewing the geometry of $K(N)$ and the correspondence between minimal Lagrangian submanifolds of N and certain Special Lagrangian submanifolds of $K(N)$.

Let $K(N)$ be the total space of the canonical bundle of N and $\pi : K(N) \mapsto N$ be the projection. There is a canonical $(n, 0)$ -form ρ on $K(N)$ defined by $\rho(a)(v_1, \dots, v_n) = a(\pi_*(v_1), \dots, \pi_*(v_n))$, $a \in K(N)$. The form $\varphi = d\rho$ is a holomorphic volume form on $K(N)$. If z_1, \dots, z_n are local coordinates on N then $(z_1, \dots, z_n, y = dz_1 \wedge \dots \wedge dz_n)$ are coordinates on $K(N)$ and $\rho = y dz_1 \wedge \dots \wedge dz_n$, $\varphi = dy \wedge dz_1 \wedge \dots \wedge dz_n$.

There is a canonical radial vector field Y on $K(N)$, given at a point $m \in K(N)$ by the vector m (viewed as a tangent vector to the linear fiber over $\pi(m)$). We have $i_Y \rho = 0$. Also the Lie derivative $\mathcal{L}_Y \rho = \rho$. So $\rho = i_Y d\rho = i_Y \varphi$. So $\mathcal{L}_Y \varphi = d(i_Y \varphi) = d\rho = \varphi$.

If N is a Kahler-Einstein manifold with positive scalar curvature then $K(N)$ has a Ricci-flat Kahler metric on it (see [7], p.108). The metric is constructed as follows : The connection on $K(N)$ induces a horizontal distribution for the projection π , with a corresponding splitting of the tangent bundle of $K(N)$ into horizontal and vertical distributions. We can identify the horizontal space at each point $m \in K(N)$ with the tangent space to N at $\pi(m)$. Let $r^2 : K(N) \mapsto \mathbb{R}_+$ be the square of the length of an element in $K(N)$ and $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a positive function with a positive first derivative. We define the metric ω_u on $K(N)$ as follows: We put the horizontal and the vertical distributions to be orthogonal. On the horizontal distribution we define the metric to be $u(r^2)\pi^*(\omega)$ and on the vertical distribution we define it to be $t^{-1}u'(r^2)\omega^*$. Here ω is the Kahler-Einstein metric on N , t is its scalar curvature and ω^* is the induced metric on the linear fibers of π . The Kahler-Einstein condition on N ensures that the corresponding 2-form ω_u defining this metric on $K(N)$ is closed, i.e. the metric is Kahler. If we take $u(r^2) = (tr^2 + l)^{\frac{1}{n+1}}$ for some positive constant l (see [7], p.109), then ω_u is complete and Ricci-flat (the Calabi metric). From now on we study $K(N)$ endowed with this metric ω_u .

We begin with the following observation : Let L be an oriented Lagrangian submanifold of N . For any point $l \in L$ there is a unique element κ_l in the fiber of $K(N)$ over l which restricts to the volume form on L . Various κ_l give rise to a section κ of $K(N)$ over L . Consider a submanifold $L^K \subset K(N)$ given by

$$L^K = ((m|m = a\kappa_l \text{ for } l \in L, a \in \mathbb{R}))$$

We have the following:

Lemma 1 *L is a minimal Lagrangian submanifold of N iff L^K is a Special Lagrangian submanifold of $K(N)$*

Here by a minimal submanifold we mean a submanifold, which is critical for the volume functional (i.e. the trace of the second fundamental form vanishes).

Proof : First we note that L^K is Special, i.e. $Im\varphi|_{L^K} = 0$. Indeed one easily verifies that $Im\rho|_{L^K} = 0$, hence $Im\varphi|_{L^K} = 0$.

We now prove that L^K is Lagrangian with respect to ω_u iff L is minimal. Let m be a point on $L^K - L$, $l = \pi(m)$ and $m = a\kappa_l$. The tangent space of L^K at m is spanned by κ_l (viewed as a vertical vector in $T_m K(N)$) and vectors $(e + a\nabla_e \kappa)$. Here e is any tangent vector to L at l (viewed as an element of the horizontal distribution of $T_m K(N)$) and $a\nabla_e \kappa$ lives in the vertical distribution of $T_m K(N)$. To compute $\nabla_e \kappa$ take an orthonormal frame (v_j) of $T_l L$ and extend it to an orthonormal frame of L in a neighbourhood U of l in L s.t. $\nabla^L v_i = 0$ at l (here ∇^L is the Levi-Civita connection of L). We get that

$$\nabla_e \kappa = \kappa \cdot \nabla_e \kappa(v_1, \dots, v_n) = \kappa(e(\kappa(v_1, \dots, v_n)) - \Sigma \kappa(v_1, \dots, \nabla_e v_j, \dots, v_n))$$

Now $e(\kappa(v_1, \dots, v_n)) = 0$. Also clearly

$$\kappa(v_1, \dots, \nabla_e v_j, \dots, v_n) = i \langle \nabla_e v_j, Jv_j \rangle = i \langle \nabla_{v_j} e, Jv_j \rangle = i \langle -e, J(\nabla_{v_j} v_j) \rangle$$

Here J is the complex structure on N . Thus we get that

$$a\nabla_e \kappa = -ia(Jh \cdot e)\kappa_l$$

Here $h = \Sigma \nabla_{v_j} v_j$ is the trace of the second fundamental form of L . From this one easily deduces that L^K is Lagrangian iff $h = 0$, i.e. L is minimal. Q.E.D.

The manifold L^K is invariant under the flow of the vector field Y on $K(N)$ (which is just scaling of $K(N)$ by real numbers). Vice versa we have the following:

Lemma 2 *Let L' be a Special Lagrangian submanifold of $K(N) - N$, invariant under the flow of Y . Then $L = \pi(L')$ is an (immersed) minimal Lagrangian submanifold of N .*

Proof: Let $m \in L' - N$. Since L' is Lagrangian and Y is in the tangent space $T_m L'$ then the tangent space to L' at m clearly decomposes as

$$T_m L' = \text{span}(Y) \oplus T'$$

where T' is in the horizontal distribution at m . The space $\pi_*(T')$ can be viewed as a tangent space to L at $l = \pi(m)$. Clearly this tangent space $T_l L$ is Lagrangian, i.e. L is Lagrangian. Also L' was Special and we have seen in the beginning of this section that $i_Y \varphi = \rho$. Thus m (viewed as an $(n, 0)$ -form on N at l) restricts to a real n -form on $T_l L$, i.e. $m \in L^K$. Hence locally L' coincides with L^K . From Lemma 1 we deduce that L is minimal. Q.E.D.

2.2 SLaG submanifolds on $K(N)$ via symplectic reduction

In the previous section we showed how to find minimal Lagrangian submanifolds of N from certain SLaG submanifolds of $K(N)$. In this section we will see that

if we have a torus action on N then we can find T -invariant SLAG submanifolds of $K(N)$ from SLAG submanifolds of a certain symplectic reduction of $K(N)$.

Let T^k act on N . Then this action induces a T^k -action on $K(N)$. Let \mathcal{T} be the Lie algebra of T , $v \in \mathcal{T}$, X_v be the flow vector field on N and X'_v the flow vector field on $K(N)$. So $\pi_*(X'_v) = X_v$. Let $l \in N$ and $m \in K_l = \pi^{-1}(l)$. Let $R(m)$ be the vertical part of X'_v at m . Since $R(m)$ is vertical, it can be viewed as an element of K_l . The correspondence $m \mapsto R(m)$ is a linear correspondence on K_l . Hence there is a complex number $\sigma_l(v)$ s.t. $R(m) = \sigma_l(v)m$. At a regular point l of the T -action $\sigma_l(v)$ can also be found in a following way : Take any unit length element $\xi \in K_l$. Extend ξ along the orbit of X_v to be invariant under the flow of X_v . Then one easily computes that $\sigma_l(v) = \nabla_{X_v} \xi \cdot \xi$. Since the flow of X_v is given by holomorphic isometries, ξ has unit length. Hence $\sigma_l(v)$ is purely imaginary. Also $\sigma_l(v)$ is linear in v (because $R(m)$ is given by the vertical part of the differential of the T -action at m , and this differential is a linear map from \mathcal{T} to $T_m K(N)$). Hence $i\sigma$ can be viewed as a map from N to the dual Lie algebra \mathcal{T}^* . This map is T -invariant.

Let $t > 0$ be the scalar curvature of N .

Lemma 3 *The map $\mu = -it^{-1}\sigma$ is a moment map for the action.*

Proof: Let $v \in \mathcal{T}$. We need to show that $d(-it^{-1}\sigma(v)) = i_{X_v}\omega$. We will do it at a regular point l of the action. Choose any unit length element ξ of $K(N)$ over l . We can extend ξ to be a local unit length section, invariant under the X_v -flow. ξ defines a connection 1-form ψ , $\psi(u) = \nabla_u \xi \cdot \xi$. ψ is invariant under the X_v -flow and the K-E condition says that $id\psi = t\omega$. So

$$0 = \mathcal{L}_{X_v}\psi = d(i_{X_v}\psi) + i_{X_v}d\psi = d\sigma(v) - it(i_{X_v}\omega)$$

So μ is a moment map. Q.E.D.

Remark: By the construction of μ we get that $\mu(v) = 0$ for some $v \in \mathcal{T}$ at a point $l \in N$ iff the vector field X'_v is horizontal at $\pi^{-1}(l)$.

Lemma 4 *The map $\mu' = u\pi^{-1}(\mu)$ is a moment map for the T -action on $K(N)$.*

Proof: Let $v \in \mathcal{T}$. We need to prove that $d\mu'(v) = i_{X'_v}\omega_u$.

We will study ω_u in more detail (see [7]). Let $m \in N$ be a regular point for the T^n -action and ξ a unit length element of $K(N)$ over m . We can extend ξ to be a local unit length section of $K(N)$, invariant under the flow of X_v . ξ gives rise to a connection 1-form ψ for the connection on $K(N)$ and the Einstein condition tells that $id\psi = t\omega$. The section ξ defines a complex coordinate a on $K(N)$, which is invariant under the X'_v -flow. Also the form $b = da + a\pi^*\psi$ vanishes on the horizontal distribution (see [7], p. 108). We have $r^2 = a\bar{a}$ and $u = u(r^2)$. Also the Kahler form ω_u on $K(N)$ is given by

$$\omega_u = u\pi^*\omega - it^{-1}u'b \wedge \bar{b}$$

One directly verifies that $\omega_u = d\eta$ for $\eta = it^{-1}u\pi^*\psi - it^{-1}\frac{u d\bar{a}}{\bar{a}}$. By our construction the flow of X'_v leaves η invariant. So

$$0 = \mathcal{L}_{X'_v}\eta = i_{X'_v}d\eta + d(i_{X'_v}\eta) = i_{X'_v}\omega_u + d(it^{-1}u\psi(X_v)) = i_{X'_v}\omega - d(\mu'(v))$$

Here we used the fact that $d\bar{a}(X'_v) = 0$ and $\psi(X_v) = \sigma(v)$. So μ' is a moment map and we are done. Q.E.D.

Let now L' be a (connected) SLAG submanifold of $K(N)$, invariant under the T -action and under the Y -flow. Since L' is Lagrangian and T -invariant, the moment map μ' is constant on L' . But $\mu' = u\pi^{-1}(\mu)$ and $Y(\mu') = 2r^2u'\pi^{-1}(\mu)$. So we have $\pi^{-1}(\mu) = 0$ on L' . Let Z be the zero set of μ . Then $L' \subset Z' = \pi^{-1}(Z) = \mu'^{-1}(0)$.

Let $Z'' = Z' - Z$. From now on we assume that T acts freely on Z'' (we will demonstrate examples where this holds in Section 3). We have a symplectic reduction $N_{red} = Z/T$ and a (smooth) symplectic reduction $Q = Z''/T$, endowed with a Kahler metric ω' .

Let v_1, \dots, v_k be a basis for \mathcal{T} and X'_1, \dots, X'_k be the corresponding flow vector fields on $K(N)$. Let $\varphi^* = i_{X'_1} \dots i_{X'_k} \varphi$ be an $(n-k, 0)$ -form on $K(N)$, obtained by contracting φ by X'_1, \dots, X'_k . Let $\rho^* = i_{X'_1} \dots i_{X'_k} \rho$. We claim that

$$\varphi^* = (-1)^k d\rho^*$$

We prove this by induction on k . Namely let $\varphi_l^* = i_{X'_1} \dots i_{X'_l} \varphi$ and $\rho_l^* = i_{X'_1} \dots i_{X'_l} \rho$. We claim that $\varphi_l^* = (-1)^l d\rho_l^*$. For $l = 1$ we have that ρ is X'_1 -invariant. Hence

$$0 = \mathcal{L}_{X'_1} \rho = d\rho_1^* + \varphi_1^*$$

Now we use induction. The form ρ_{l-1}^* is X'_l -invariant. Hence

$$0 = \mathcal{L}_{X'_l} \rho_{l-1}^* = d\rho_l^* + (-1)^{l-1} \varphi_l^*$$

and we are done by induction.

Both φ^* and ρ^* are T -invariant. Let $\nu : Z \mapsto N_{red}$ and $\nu' : Z'' \mapsto Q$ be the quotient maps. One easily sees that there is a unique $(n-k, 0)$ -form φ' on Q and a unique $(n-k-1, 0)$ -form ρ' on Q s.t.

$$\nu^*(\varphi') = \varphi^*, \quad \nu^*(\rho') = \rho^*, \quad \varphi' = (-1)^k d\rho'$$

We can define on Q SLAG submanifolds L'' by the conditions $\omega'|_{L''} = 0$, $Im\varphi'|_{L''} = 0$.

The vector field Y is tangent to Z'' and T -invariant, hence it projects to a vector field Y' on Q . We had $i_Y \varphi = \rho$ on $K(N)$. Hence we also have $i_{Y'} \varphi' = \rho'$ on Q . We obviously have the following:

Lemma 5 *Let L'' be a SLAG submanifold of Q , invariant under Y' . Then $L' = \nu'^{-1}(L'')$ is a SLAG submanifold of $K(N)$, invariant under T^k and under the Y -flow.*

The proof of the lemma is obvious.

2.3 Complexity one actions and periodic orbits

In the previous section we have shown that one can reduce the problem of finding T and Y -invariant SLAG submanifolds of $K(N)$ to finding Y' -invariant

SLag submanifolds of Q . In this section we will assume that $k = n - 1$. Let $X \subset Z''$ be the set of elements in Z'' of unit length and $S = X/T \subset Q$. We will show that there is a vector field W on S s.t. there is correspondence between Y' -invariant SLag submanifolds of Q and the trajectories of the W -flow on S .

As we saw the tangent bundle of $K(N)$ decomposes as a direct sum $V \oplus H$ of the vertical and the horizontal distributions. Let U be the image of the Lie algebra of T under the differential of the action on $K(N)$. At points of Z'' U is an $(n - 1)$ -dimensional vector space, and it is contained in the horizontal distribution H (since on $\pi(Z'')$ the moment map μ vanishes). Also the Kahler form ω_u restricts to 0 on U . Let U^c be the complexification of U in the tangent bundle to $K(N)$. Then U^c can be viewed as a complex $(n - 1)$ -dimensional vector bundle over Z'' . Let H' be the orthogonal complement of U^c in H . Then the tangent bundle of Z'' is a direct sum $V \oplus H' \oplus U$. Also the quotient of $V \oplus H'$ under the T -action can be identified with the tangent bundle to the symplectic reduction $Q = Z''/T$. Since H' and V are T -invariant the tangent bundle to Q splits as a direct sum of 2 complex line bundles: $TQ = V \oplus H'$. Also V and H' are orthogonal both with respect to the symplectic form ω' and the Riemannian metric on Q .

There is a natural circle action on X (given by the multiplication by complex numbers of absolute value 1 on $K(N)$). This action is T -invariant, hence it induces a circle action on $S = X/T$. Let F be the vector field generating this action on S . Then $F = J(Y')$ (here J is the complex structure on Q). Also both Y' and F are in the vertical distribution V along S and the tangent bundle TS of S splits as a direct sum $TS = H' \oplus \text{span}(F)$.

Let γ be some path in S and let γ^Q be the orbit of γ under the Y' -flow in Q . We wish to understand when γ^Q is a SLag submanifold of Q . Let W be a tangent vector to γ . Clearly for γ^Q to be Lagrangian we need W to live in the horizontal distribution H' . The form $\rho' = i_{Y'}\varphi'$ is a (non-zero) $(1,0)$ -form on H' . Hence the form $\text{Im}\rho'$ has a 1-dimensional kernel in H' . Clearly for γ^Q to be Special we need W to belong to this kernel. We can normalize W s.t. $\text{Re}\rho'(W) = 1$. Those conditions give rise to a non-vanishing horizontal vector field W on S . Let γ be a trajectory of W on S and consider $\gamma^Q \subset Q$. The forms ω' and φ' vanish on γ^Q along γ . Also the Y' -flow preserves the horizontal distribution and $\mathcal{L}_{Y'}\rho' = \rho'$. From this we easily deduce that γ^Q is a Y' -invariant SLag submanifold of Q . From the above discussion we get the following:

Lemma 6 *Let γ be a trajectory of W on S . Then $L_\gamma = \pi(\nu'^{-1}(\gamma^Q))$ is an immersed minimal Lagrangian submanifold of N . If γ is periodic then L_γ is an immersed minimal Lagrangian torus.*

There is one general relation among trajectories of W , which will later be important: Consider the circle action on $K(N)$ as before. The $(n,0)$ -form ρ is equivariant with respect to this action, i.e. if $\lambda \in S^1$ then $\lambda^*(\rho) = \lambda\rho$. Since $\varphi = d\rho$ we get that φ is also equivariant with respect to this action. Thus we also deduce that ρ' and φ' are equivariant with respect to the circle action on Q . Also this action preserves the horizontal distribution H' on S . Consider an element $-1 \in S^1$. Then $-1^*(\rho') = -\rho'$. From this we deduce that the -1 -action

on S reverses the vector field W , i.e. $-1_*(W) = -W$. Thus the -1 -action sends W -trajectories to W -trajectories, but it reverses their directions.

3 Toric K-E manifolds

In Section 2.3 we saw that if we have a T^{n-1} -action on N , then one can construct minimal Lagrangian submanifolds of N from trajectories of the vector field W on S . In order to do this we needed T to act freely on Z'' . In this section we will show a class of examples where this holds. We will also investigate periodic orbits of W on S (to construct immersed minimal Lagrangian tori).

Let N be toric, i.e. we have an effective structure-preserving T^n -action on N . For recent results on toric K-E manifolds we refer the reader to [8] and [2]. We will use various $(n-1)$ -dimensional sub-tori of T to construct invariant minimal Lagrangian submanifolds. But first we will see that there is a unique minimal Lagrangian torus, invariant under the whole of T .

Suppose L is a regular orbit of the T -action, which is a minimal submanifold. Then L^K is a SLAG submanifold of $K(N)$. The moment map μ' of $K(N)$ is constant on L^K . As we have seen in Section 2.2, we must have $\mu = 0$ on L i.e. $L \subset \mu^{-1}(0)$. By Atiyah's result [1], $\mu^{-1}(0)$ is connected, hence $L = \mu^{-1}(0)$. So if a regular orbit, which is a minimal submanifold, exists, it must coincide with $\mu^{-1}(0)$. Next we prove that such an orbit does exist.

Lemma 7 *Let (M^{2n}, ω) be a compact symplectic manifold and g some Riemannian metric on M . Suppose that we have an effective Hamiltonian n -torus action on M , which preserves g . Then there is a regular orbit of the action, which is a minimal submanifold with respect to g . In fact this orbit maximizes volume among the orbits.*

Proof: We have a moment map μ and smooth orbits are levels set of the moment map. For a regular orbit L to be a minimal submanifold, it is obviously necessary to be a critical point for the volume functional on the orbits. We note that it is also sufficient. Indeed let v be any element of the Lie algebra \mathcal{T} of the torus T^n . Then $\mu(v)$ is T^n -invariant, and so is the gradient $\nabla\mu(v)$. Also this gradient is orthogonal to the orbits. Consider now this gradient flow. It commutes with the T^n -action, hence it sends orbits to orbits. Since L is critical for the volume functional on the orbits, we get from the first variation formula $\int_L h \cdot \nabla\mu(v) = 0$. Here h is a trace of the second fundamental form of L . But both h and $\nabla\mu(v)$ are T^n -invariant, hence we are integrating a constant. So $h \cdot \nabla\mu(v) = 0$ pointwise. Now v was arbitrary, hence $h = 0$.

We want to find a regular orbit, which is maximum point for the volume functional on the orbits. First we need to prove that the volume functional is continuous on the space of orbits. Let L' be a regular orbit for the torus action. Then the differential of the moment map is surjective along L' . From this one easily deduces that orbits of the action near L' coincide with level sets of the moment map. So obviously the volume functional is continuous near L' . Next

we prove that the volume functional is continuous near the singular orbits. This follows from the following easy Lemma:

Lemma 8 *Let L be an orbit with a positive dimensional stabilizer $T' \subset T$ and $x \in L$. Then for any $\epsilon > 0$ there is a neighbourhood U of x s.t. any orbit passing through U has volume $< \epsilon$.*

Proof: Take a (unit) vector e_1 in the Lie Algebra of T' . Then the corresponding flow vector field X_1 vanishes along L . Extend e_1 to an o.n. basis e_2, \dots, e_n of \mathcal{T} . The flow vector fields X_i will have uniformly bounded lengths. We choose a neighbourhood U of x in which X_1 has sufficiently small length and it is clear that volumes of the orbits through U will be sufficiently small. Q.E.D.

So the volume functional is continuous on the space of orbits and we can find an orbit L , which maximizes volume among the orbits. Obviously L must be a regular orbit (since singular orbits have zero volume). As we have seen, L is a minimal submanifold of N and we are done. Q.E.D.

Let now $T'' \subset T^n$ be some $(n-1)$ -torus in T and let μ'' be the canonical moment map for the T'' -action on N as in Section 2.2. Then μ'' is just the restriction of μ to the dual Lie algebra of T'' . In order to apply the constructions of Section 2.3 we want T'' to act freely on Z'' . The following lemma guarantees the existence of such T'' :

Lemma 9 *Let N^{2n} be a K-E manifold with an effective T^n -action as above. Then there is an $(n-1)$ -torus $T'' \subset T$ s.t.*

- i) The differential of the T'' -action on N is injective along Z and T acts freely on Z'' .*
- ii) There is an element v in the Lie algebra of T'' s.t. the flow vector field X_v doesn't have a constant length along Z .*

Remark: Condition ii) in the lemma will be used to show that certain minimal Lagrangian tori we shall construct have Killing fields of non-constant length, hence they are not flat.

Proof: Let $T'' \subset T^n$ be some $(n-1)$ -torus. First we prove that if the differential of the T'' -action on N is injective along Z , then the T -action on Z'' is free. Suppose not. Then there is a point $l \in Z''$ and an element $1 \neq t \in T$ s.t. $t \cdot l = l$. In that case t also preserves the points on the T'' -orbit through l . The tangent space P to this orbit at l is in the horizontal distribution at l (since we are at the zero set of the moment map μ'). Also $\omega_u|_P = 0$. So the differential dt of the t -action at l acts trivially on the complexification P^c of P . Also dt acts trivially on the vertical distribution $V(l)$ at l . The vector space $P^c \oplus V(l)$ is a complex vector space of dimension n and dt acts trivially on it. Also dt preserves the holomorphic volume form φ at l . Hence dt is trivial at l . Hence t acts trivially on $K(N)$ and on N , but the T -action on N was effective—a contradiction.

Next we wish to understand for which $(n-1)$ -tori $T'' \subset T$ the differential of the T'' -action is injective along the zero set Z of the canonical moment map of T'' . Let \mathcal{T}^* be the dual Lie algebra of T and let $\Lambda \subset \mathcal{T}^*$ be the weight lattice

of T . Any element $0 \neq v \in \Lambda$ defines an $(n-1)$ -torus $T_v \subset T$ s.t. v vanishes on the Lie algebra of T_v . Let μ be the canonical moment map of T and μ_v be the canonical moment map of T_v . Then μ_v is just the restriction of μ to the dual Lie algebra of T . It is therefore clear that μ_v vanishes at a point $n \in N$ iff $\mu(n)$ is proportional to v . Since N is a toric variety, the moment polytope is convex and has no faces in the interior. Since 0 is in the interior of the moment polytope, it is clear that $Z = \mu^{-1}[t_1 v, t_2 v]$ with $t_1 < 0 < t_2$. For any $t_1 < t < t_2$ the value tv is in the interior of the moment polytope, while $t_1 v$ and $t_2 v$ are not.

Suppose the line $\text{span}(v)$ doesn't intersect any of the $(n-2)$ -faces of the moment polytope. This means that any point in Z has either a trivial or a 1-dimensional stabilizer in T . We claim that the differential of the T'' -action is injective along Z . Suppose not. Then there is a point $n \in Z$ and a vector $0 \neq w$ in the Lie algebra of T'' s.t. the flow vector field X_w vanishes at n . Since $n \in Z$ the flow vector field X'_w of w on $K(N)$ is horizontal along $\pi^{-1}(n) \subset K(N)$, hence it vanishes along $\pi^{-1}(n)$. Let $g = \exp(tw)$ for some $t \in \mathbb{R}$. Then the g -action on $\pi^{-1}(n)$ is trivial. But this means that the differential dg of the g -action on the tangent space $T_n N$ has Jacobian 1. Also g acts trivially on the orbit L' of the T -action through n . The tangent space $T_n L'$ of L' at n is $(n-1)$ -dimensional and ω restricts to 0 on it. Hence its complexification $T_n L'^c$ is a complex $(n-1)$ -dimensional space and dg acts trivially on it. Also dg has Jacobian 1. Hence dg is trivial, hence g acts trivially- a contradiction.

A generic line in the projective space PT^* doesn't intersect the $(n-2)$ -faces of the moment polytope of μ . Also the set of lines passing through points of Λ is dense in PT^* . So we can easily find $v \in \Lambda$ so that i) holds for T_v . In order to ensure that ii) holds, consider a point b in the $(n-2)$ -face of the moment polytope. The orbit $\mu^{-1}(b)$ has a stabilizer of dimension at least 2. Hence we can find a vector $0 \neq w \in \mathcal{T}$ s.t. $b(w) = 0$ and the flow vector field X_w vanishes along $\mu^{-1}(b)$. We can find a sequence of elements $v_k \in \Lambda$ s.t. the lines $(v_k) = \text{span}(v_k)$ do not intersect the $(n-2)$ -faces of the moment polytope and (v_k) converge to the line $(b) = \text{span}(b)$ in PT^* . We can also find a sequence of vectors $w_k \in \mathcal{T}$ s.t. $v_k(w_k) = 0$ and $w_k \mapsto w$.

Each v_k defines an $(n-1)$ -torus $T_k \subset T$. Let μ_k be the canonical moment map of T_k , and Z_k be the zero set of μ_k . We can find points n_k on Z_k s.t. n_k converge to a point $n \in \mu^{-1}(b)$. Let X_k be the flow vector field of w_k . Then the length of X_k at points n_k goes to 0 as $k \mapsto \infty$. On the other hand the torus $L = \mu^{-1}(0)$ is contained in all of Z_k . Moreover the lengths of X_k along L are a-priori bounded from below. So we deduce that for k large enough the torus $T'' = T_k$ satisfies the conditions i) and ii) of the lemma. Q.E.D.

From now on we pick a sub-torus $T'' \subset T$ satisfying the conditions of Lemma 9. We can use the results of Section 2.3 to deduce that one can construct minimal Lagrangian submanifolds of N from the trajectories of the vector field W on S . From Lemma 6 we deduce that in order to obtain immersed minimal Lagrangian tori we need the orbits to be periodic. A first step in finding such orbits will be the following observation: The circle $R = T/T''$ acts freely on Q and on S . Let $w \neq 0$ be some element in the Lie algebra of R . We have the flow vectors field A_w for the w -action on Q and the vector fields A_w and W commute. We also have

a $(1, 0)$ -form ρ' and a holomorphic $(2, 0)$ -form φ' on Q with $\varphi' = (-1)^{n-1}d\rho'$. The flow of A_w preserves ρ' and φ' . A key point in finding periodic trajectories of W is the fact that there is a function on S constant along the trajectories:

Lemma 10 *Let $h = \rho'(A_w)$ and $f = \text{Re}(h)$. Then f is constant along the trajectories of W .*

Proof: We have

$$0 = \mathcal{L}_{A_w}\rho' = d(i_{A_w}\rho') + i_{A_w}d\rho' = dh + (-1)^{n-1}i_{A_w}\varphi'$$

So $dh = (-1)^n i_{A_w}\varphi'$. So $dh(W) = (-1)^n \varphi'(A_w, W)$. The vector field A_w is in the tangent bundle to S , hence we can decompose it into $A_w = A_w^H + \lambda F$. Here A_w^H is the horizontal part of A_w (i.e. the part in the distribution H'), F is the generator of the S^1 -action on S and $\lambda \in \mathbb{R}$ (see Section 2.3). W is horizontal and H' is a 1-dimensional complex vector bundle. The form φ' is a $(2, 0)$ -form on Q . Hence $\varphi'(A_w^H, W) = 0$. Also $F = JY'$. Hence $\varphi'(F, W) = i\varphi'(Y', W)$. By the construction of W we had that $\varphi'(Y', W)$ is real. From all this we deduce that $dh(W)$ is purely imaginary, hence $df(W) = 0$, i.e. f is constant along the trajectories of W . Q.E.D.

From the previous lemma we deduce that the trajectories of W live on level sets of the function f . We need to understand those level sets in more detail.

We had our symplectic reductions $N_{red} = Z/T''$ and Q and we have a natural projection $\pi' : Q \mapsto N_{red}$. Let v be an element of the weight lattice Λ of \mathcal{T}^* defining the torus T'' . As we have seen Z is equal to $\mu^{-1}[t_1 v, t_2 v]$ for $t_1 < 0 < t_2$. T'' acts freely on $Z_0 = \mu^{-1}(t_1 v, t_2 v)$ and we have $N_0 = Z_0/T'' \subset N_{red}$, which is the smooth part of N_{red} . We also have 2 points $a_i = \mu^{-1}(t_i v)/T'' \in N_{red}$ and N_{red} is a disjoint union of a_1, a_2 and N_0 . We have $S_0 = \pi'^{-1}(N_0) \cap S$, and S_0 is a fiber bundle over N_0 with fibers being the orbits of the S^1 -action on S (see Section 2.3). This action is free on S_0 . Also each $K_i = \pi'^{-1}(a_i)$ is an orbit of the S^1 -action on S , but this action on each K_i might have a finite stabilizer.

We have seen in Section 2.3 that the form ρ' is equivariant with respect to the S^1 -action on S . The flow vector field A_w is invariant under S^1 -action. Hence the function $h = \rho'(A_w)$ is S^1 -equivariant.

On Z_0 we had an oriented Lagrangian distribution D , given by the image of \mathcal{T} under the differential of the action on N . This distribution gives rise to a unit length section κ of $K(N)$ over Z_0 as in Lemma 1. This section is T -invariant, hence it gives rise to an R -invariant section κ' of S_0 over N_0 . By definition κ restricts to a positive real n -form on the distribution D . From this we deduce that $h = \rho'(A_w)$ is real and positive along κ' .

We can normalize w s.t. $v(w) = 1$. We have a function $\tau = \mu(w)$ on N_{red} , and the image of this function is the interval $[t_1, t_2]$. For each $t_1 \leq t \leq t_2$ the level set $\tau^{-1}(t)$ is an orbit of the R -action on N_{red} . Let $L' = \tau^{-1}(0)$, $L_+ = \kappa'(L') \subset S$ and $L_- = (-1) \cdot L_+$. Each L_\pm is an orbit of the R -action on S . Also at points of L_\pm the vector field A_w is horizontal (since $\mu(w) = 0$) and $\rho'(A_w)$ is real. The vector field W also satisfies those properties, hence W is proportional to A_w along L_\pm . So we see that L_\pm are trajectories W (of course

the minimal Lagrangian torus of N coming from these trajectories is the torus $L = \mu^{-1}(0)$. We have the following:

Lemma 11 *The differential df of f is non-vanishing on $S - (L_- \cup L_+)$.*

Proof: We have seen in the proof of Lemma 10 that $dh = (-1)^n i_{A_w} \varphi'$. On $S \cap \pi'^{-1}(N_{red} - L')$ the vertical part of the vector field A_w doesn't vanish. Hence the form $i_{A_w} \varphi'$ restricts as a non-vanishing $(1, 0)$ -form on the horizontal distribution H' . From this it is clear that $df|_{H'} \neq 0$.

On $S \cap \pi'^{-1}(L') - (L_- \cup L_+)$ h is not real. Also h is equivariant with respect to the S^1 -action. Let F be the vector field generating the S^1 -action as before. Then the derivative of $f = Reh$ is non-zero in the direction of F . Q.E.D.

f attains a constant value f_+ along L_+ and a value $f_- = -f_+$ along $L_- = (-1) \cdot L_+$. Since S is compact and connected, it is clear from Lemma 11 that f_+ is the absolute maximum of f , attained only at L_+ , and f_- is the absolute minimum of f , attained only at L_- . Also for any $s \in (f_-, f_+)$, the level set $\Sigma_s = f^{-1}(s)$ is smooth.

We will also need the fact that $f|_{K_i} = 0$. To prove that we note that along K_i A_w is vertical. Indeed the action of $\exp(tw)$ on a_i is trivial for any $t \in \mathbb{R}$. Hence the action of $\exp(tw)$ on S preserves the fiber $K_i = \pi'^{-1}(a_i)$, so the vector field A_w is tangent to K_i , i.e. vertical along K_i . But from this we deduce that $h = \rho'(A_w) = 0$ at K_i , and so $f = 0$ at K_i .

Let now $\Phi = f^{-1}(f_-, f_+)$. Take any point $m \in \Phi$ and consider the level set Σ_s of f passing through m . Let Σ_s^0 be the connected component of Σ_s containing m . The vector field W is tangent to Σ_s^0 . We have a free R -action on Σ_s^0 , and this action preserves the vector field W . Also A_w is transversal to W at all points of Σ_s^0 . Indeed let $m' \in \Sigma_s^0$. If $m' \in S \cap \pi'^{-1}(N_{red} - L')$, then the vector field A_w is not horizontal, so it can't be proportional to W . If $m' \in S \cap \pi'^{-1}(L') - (L_- \cup L_+)$, then $h = \rho'(A_w)$ is not real, while $\rho'(W)$ is real. So again A_w and W can't be proportional. Thus we get that the quotient of Σ_s^0 by the R -action is a circle and W projects to a non-vanishing vector field on it. From this we deduce that the W -trajectory starting at m will intersect the R -orbit of m . Suppose it intersects this orbit for the first time at a point $\xi(m)m$, $\xi(m) \in R$. This gives rise to a well-defined function $\xi : \Phi \mapsto R$. Clearly ξ is continuous, R -invariant and constant along the trajectories of W . Also we have seen in Section 2.3 that the -1 -action on S sends W -trajectories to W -trajectories in the reverse direction. From this we easily deduce that

$$\xi(-1 \cdot m) = \xi(m)^{-1}$$

Obviously the trajectory through m is periodic iff $\xi(m)$ is a root of unity in R . Let R' be the set of roots of unity in R . Since ξ is continuous, $\xi^{-1}(R')$ will be everywhere dense in Φ unless ξ assumes a constant value not in R' on some open subset of Φ . The next lemma shows that it is impossible:

Lemma 12 *Suppose that ξ is constant on some open set $U \subset \Phi$. Then ξ is equal to a constant g on the whole of Φ and $g^2 = 1$.*

Proof: Let $S_+ = f^{-1}(0, f_+)$, $S_- = f^{-1}(f_-, 0)$. Thus $S_- = -1 \cdot S_+$. Suppose that ξ is constant on some open set $U \in \Phi$. Then ξ is constant on some open ball U' either in S_+ or in S_- . We can assume w.l.o.g. that $U' \subset S_+$. We note that S_+ is connected. In fact S_+ is given by

$$S_+ = (\kappa'(n)e^{i\theta} | n \in N_0, -\pi/2 < \theta < \pi/2) - (L_+ \bigcup L_-)$$

First we prove that ξ is a constant g on S_+ . Let A_w^H be the horizontal part of the vector field A_w . Since $S_+ \subset \pi^{-1}(N_0)$ we deduce that A_w^H doesn't vanish on S_+ . We also note that A_w^H cannot be proportional to JW . Indeed suppose that $A_w^H = \lambda JW$ for some $\lambda \in \mathbb{R}$ at some point $m \in S_+$. But then

$$h(m) = \rho'(A_w) = \rho'(A_w^H) = i\lambda\rho'(W)$$

So $h(m)$ is purely imaginary, hence $f(m) = 0$ - a contradiction. Since both A_w^H and W lie in H' , which is a complex 1-dimensional distribution, we deduce that we can find a function $b : S_+ \mapsto \mathbb{R}$ s.t. the vector fields $W' = A_w^H + bJA_w^H$ and W are proportional. Hence the trajectories of W' and W coincide. We will use W' instead of W to prove that ξ is constant on S_+ . Suppose that for a point $m \in S_+$ it takes time $t(m)$ for the W' -flow to hit the R -orbit of m . We have the following:

Lemma 13 $\xi(m) = \exp(t(m)w)$

Proof: Let γ be the trajectory of W' through m and $\gamma' = \pi'(\gamma)$ be the corresponding path in N_0 . We have an R -action on N_0 , and the corresponding flow vector field B_w for the w -flow on N_0 . We obviously have $\pi'_*(A_w^H) = B_w$. Hence the tangent field to γ' is $B_w + bJB_w$.

The R -action on N_0 is Hamiltonian with the moment map $\tau = \mu(w)$. Also the B_w -flow on N_0 commutes with the complex structure J on N_0 . Hence the vector fields B_w and JB_w commute. Let $P_1 = \gamma'(0)$ and $P_2 = \gamma'(t(m))$. Then $P_2 = \xi(m)P_1$. We need to prove that $P_2 = \exp(t(m)w)P_1$ and since the R -action on N_0 is free we would be done.

Let $\exp(xJB_w)$ be the time x flow of B_w . Note that the JB_w -flow is not complete. In fact we have

$$JB_w(\tau) = \omega_{red}(JB_w, B_w) = |B_w|^2 > 0$$

So τ increases on the JB_w -trajectories. Let $c(r) = \int_{(0,r)} b(t)dt$ for $0 \leq r \leq t(m)$. Consider a path $\gamma''(r) = \exp(c(r)(JB_w))\exp(rw)(P_1)$ (note that we flow P_1 with respect to rB_w first). Then $\gamma''(r)$ is defined for small values of r . Also the tangent vector to γ'' is $B_w + b(r)JB_w$. So γ'' coincides with γ' whenever it is defined.

Suppose on γ' τ ranges between s_1 and s_2 . Then $t_1 < s_1$ and $s_2 < t_2$. Pick any r for which $\gamma''(r)$ is defined and consider the path $\exp(tJB_w)\exp(rw)(P_1)$ for t ranging between 0 and $c(r)$. The function τ is increasing along the path, and on the endpoints it's values are between s_1 and s_2 . Hence this path lives

in the compact set $A = \tau^{-1}[s_1, s_2]$ in N_0 . From this one can easily deduce that $\gamma''(r)$ is well defined for all $0 \leq r \leq s$ and coincides with $\gamma'(r)$. In particular $P_2 = \exp(c(t(m))JB_w)\exp(t(m)w)(P_1)$. Now

$$\tau(P_2) = \tau(P_1) = \tau(\exp(t(m)w)P_1)$$

and τ increases on the trajectories of JB_w . So we get that $c(t(m)) = 0$, i.e. $P_2 = \exp(t(m)w)P_1$. Q.E.D.

Now we can prove that ξ is constant on S_+ . Since ξ is constant on U' we get that $t(m)$ is a constant t on U' . Let ϕ_t be the time t flow of W' on S_+ . Consider the map $\chi = \exp(-tw) \cdot \phi_t : S_+ \mapsto S_+$. S_+ is a connected real analytic manifold and χ is a real analytic map. Also χ is the identity map on U' . So we deduce that χ is the identity map. So ϕ_t is the multiplication by $g = \exp(tw)$ on S_+ . From this we easily deduce that $\xi = g$ on S_+ .

So ξ assumes a constant value g on S_+ , and hence it assumes a constant value g^{-1} on $S_- = -1 \cdot S_+$. Let $\Delta = f^{-1}(0)$. Then Δ is the common boundary of S_+ and S_- in Φ . Since ξ is continuous, we must have $g = g^{-1}$, i.e. $g^2 = 1$. Q.E.D.

From this we get an immediate corollary

Corollary 1 *The set $\xi^{-1}(R')$ is everywhere dense in Φ .*

We are now ready to state and prove our main result:

Theorem 1 *Let N^{2n} be a K-E manifold with positive scalar curvature with an effective T^n -action. Then precisely one regular orbit of the action is a minimal Lagrangian submanifold of N . Moreover there is an $(n-1)$ -torus $T^{n-1} \subset T^n$ and a sequence of non-flat T^{n-1} -invariant immersed minimal Lagrangian tori $L_k \subset N$ s.t. L_k locally converge to L (in particular the supremum of sectional curvatures of L_k and the distance between L and L_k goes to 0 as $k \mapsto \infty$).*

Proof: Choose a torus $T'' = T^{n-1}$ which satisfies the conditions of Lemma 9. By Corollary 1 we can choose a sequence of points $m_k \in \xi^{-1}(R')$ s.t. m_k converge to a point m in L_+ . The W -trajectories γ_k through m_k are periodic and live on level sets Σ_k of f with Σ_k converging to L_+ . From this we easily see that γ_k locally converge to the trajectory L_+ . One easily deduces that the immersed minimal Lagrangian tori L_k which γ_k define as in Lemma 6 locally converge to the minimal, T -invariant orbit L .

Finally we prove that L_k are not flat. From Lemma 9 we get a vector v in the Lie algebra of T'' s.t. the flow vector field X_v of v doesn't have a constant length on Z . Now the vector field X_v along L_k is a Killing vector field of L_k . So to prove that L_k is not flat it is enough to prove that $|X_v|^2$ is non-constant on L_k .

The function $|X_v|^2$ is T -invariant on Z . Thus it can be viewed as an R -invariant function on N_{red} , i.e. it can be viewed as a function of $\tau = \mu(w)$ on N_{red} . Also $|X_v|^2$ is a real analytic function on $N_0 = \tau^{-1}(t_1, t_2)$. Since $|X_v|^2$ is non-constant, it is nowhere a locally constant function of τ . Since γ_k are different from L_{\pm} , we easily deduce that $\tau(\pi'(\gamma_k))$ are non-trivial intervals in (t_1, t_2) . Hence $|X_v|^2$ is non-constant on L_k and we are done. Q.E.D.

References

- [1] M. Atiyah : Convexity and commuting Hamiltonians, Bull. London Math. Soc., vol 14, no. 46, 1982
- [2] V. Batyrev and E. Selivanova : Einstein-Kahler metrics on symmetric toric Fano manifolds, J. Reine Angew Math. 512 (1999), 225-236
- [3] I. Castro, F. Urbano : New examples of minimal Lagrangian tori in the complex projective plane, Manuscripta Math. 85 (1994), no 3, 265-281
- [4] E. Goldstein : Calibrated Fibrations on complete manifolds via torus action, math.DG/0002097
- [5] R. Harvey and H.B. Lawson : Calibrated Geometries, Acta Math. 148 (1982)
- [6] M. Haskins : Special Lagrangian cones, math.DG/0005164
- [7] S. Salamon : Riemannian Geometry and Holonomy Groups, Pitman Press
- [8] G. Tian : Kahler-Einstein metrics with positive scalar curvature, Inven. Math. vol 137, 1997

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